The distributive law of tensor product over direct product

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中文摘要

在模 (module) 中，張量積(tensor product)對直和 (direct sum) 具有分配性，已爲人所證明。至於張量積對直積 (direct product)，則並不具有分配性。本論文的目的是探討分配性成立與不成立的各種情形。在第一章中，我們導引一些已知的結果。第二章中，我們提供二個模分配性不成立的例子。在第三章中，我們給予分配性成立的充分條件，並加以證明作爲結論。

0. ABSTRACT

The object of this note is to discuss the distributive law of tensor product over direct product. This property is always not true for modules. In this paper, we shall give two examples, it will show that their distributive law does not exist. We also establish a sufficient condition which will satisfy the distributive law.

I. PRELIMINARIES

In what follows, R will always be an associative ring with identity element 1, and all modules will be considered as unital modules.

In P. J. Hilton and U. Stammbach [1], the following results have already been proved:

Lemma 1.1: (i) Let [M_α], $\alpha \in J$, be a family of right R-modules and let N be a left R-module. Then

$$\left(\bigoplus_{\alpha \in J} M_\alpha \right) \otimes_R N \cong \bigoplus_{\alpha \in J} (M_\alpha \otimes_R N)$$

(ii) If $M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is an exact sequence of right R-modules, then for any left R-module N, the sequence

$$M' \otimes_R N \xrightarrow{f \otimes 1} M \otimes_R N \xrightarrow{g \otimes 1} M'' \otimes_R N \to 0$$

(where 1 denotes the identity mapping on N) is exact.

Lemma 1.2: If M is a left R-module and N a right R-module, then

$$R \otimes_R M \cong M \quad \text{and} \quad N \otimes_R R \cong N.$$

If M and N are free modules, then $M \otimes_R N$ is a free module. Thus, we have the following

Lemma 1.3: Let N be a free left R-module with base $[y_i]$, $i \in J$, and let M be an arbitrary right R-module. Then each element of $M \otimes_R N$ has a unique representation in the form

$$\sum_i (m_i \otimes y_i)$$

where m_i belongs to M and $m_i = 0$ for almost all i.

Remark: There is, of course, a similar result when M is free and N is arbitrary.

The proof of this lemma is given in Northcott [2].

Corollary 1.4: Let M be a free right R-module, then $m \otimes n = 0$ $(m \in M, n \in N)$ implies $m = 0$ or $n = 0$.

Lemma 1.5: Let R be a commutative ring, M a free R-module with basis $[x_i]$, $i \in J$, N a free R-module with basis $[y_h]$, $h \in H$. Then $M \otimes_R N$ is a free R-module with basis $[x_i \otimes y_h]$, $i \in J$, $h \in H$.

$S^{-1}R$ is the ring of fraction of R with respect to S and $S^{-1}M$ is the localization of R-module M with respect to S.

Lemma 1.6: Let M be a R-module. Then

$$S^{-1}R \otimes_R M \cong S^{-1}M .$$

The proof of this lemma is given in M.F. ATIYAH [3].

Remark: The ring of the previous lemma will be consider as a commutative ring. It is used in the proof of the following example 2.

II. TWO EXAMPLES

In this section, we shall find two examples which do not satisfy the distributive law of tensor product over direct product. First, we shall make an example such that the mapping $(\prod_\alpha M_\alpha) \otimes_R N \to \prod_\alpha (M_\alpha \otimes_R N)$ is one to one but not onto.

Example 1: Let $[\, M_\alpha \,]$, $\alpha \in J$, be a family of free R-modules, N a free R-module of infinite rank, and let $\theta : (\prod_\alpha M_\alpha) \times N \to (\prod_\alpha M_\alpha)\, O_R\, N$ be a R-balance, $\gamma : (\prod_\alpha M_\alpha) \times N \to \prod_\alpha (M_\alpha \otimes_R N)$ a R-balance such that $\theta\, ((m_\alpha)_\alpha, n) = (m_\alpha)_\alpha \otimes n$ and $\gamma\, ((m_\alpha)_\alpha, n) = (m_\alpha \otimes n)_\alpha$. Then the following diagram

$$\begin{array}{ccc} (\prod_\alpha M_\alpha) \times N & \xrightarrow{\ \theta\ } & (\prod_\alpha M_\alpha) \otimes_R N \\ & \gamma \searrow \quad \swarrow \mu & \\ & \prod_\alpha (M_\alpha \otimes_R N) & \end{array}$$

is commutative. (By tensor product definition)

There is a unique R-homomorphism $\mu : (\prod M_\alpha) \otimes_R N \to \prod_\alpha (M_\alpha \otimes_R N)$ such that $\mu\,\theta = \gamma$, that is,

$$\mu\, ((m_\alpha)_\alpha \otimes n) = (m_\alpha \otimes n)_\alpha.$$

We show that μ is monomorphism. For each $(m_\alpha)_\alpha \otimes n \in \ker \mu$, $\mu\,((m_\alpha)_\alpha \otimes n) = (m_\alpha \otimes n)_\alpha = 0$ implies $m_\alpha \otimes n = 0$ for each α. If $n = 0$ then $(m_\alpha)_\alpha \otimes n = 0$. If $n \neq 0$, by corollary 1.4, $m_\alpha = 0$ for each α then $(m_\alpha)_\alpha \otimes n = 0$. Hence μ is a monomorphism.

We claim that μ is not epimorphism. Suppose that μ is an epimorphism and $[\, x_{\alpha,i} \,]_i$ is a basis of M_α for each α, $[\, y_j \,]_j$ is a basis of N. For each α, take $x_{\alpha,1} \in [\, x_{\alpha,i} \,]_i$, $y_\alpha \in [\, y_j \,]_j$ and $y_\alpha \neq y_\beta$, if $\alpha \neq \beta$. Then $(x_{\alpha,1} \otimes y_\alpha)_\alpha \in \prod_\alpha (M_\alpha \otimes_R N)$. Since

μ is an epimophism, there is $(a_\alpha)_\alpha \otimes n \in (\prod_\alpha M_\alpha) \otimes_R N$ such that

$$\mu((a_\alpha)_\alpha \otimes n) = (x_{\alpha,1} \otimes y_\alpha)_\alpha = (a_\alpha \otimes n)_\alpha ,$$

that is, for each α, $x_{\alpha,1} \otimes y_\alpha = a_\alpha \otimes n$.

For each $a_\alpha \in M_\alpha$ $n \in N$,

$$n = b_1 y_1 + b_2 y_2 + \ldots + b_k y_k = \sum_{j=1}^{k} b_j y_j$$

$$a_\alpha = c_{\alpha,1} x_{\alpha,1} + \ldots + c_{\alpha,p} x_{\alpha,p} = \sum_{i=1}^{p} c_{\alpha,1} x_{\alpha,1}$$

then $$a_\alpha \otimes n = \sum_{j=1}^{k} \sum_{i=1}^{p} c_{\alpha,i} \; b_j \; (x_{\alpha,i} \otimes y_j) = x_{\alpha,1} \otimes y_\alpha .$$

Put $\alpha = k+1$,

$$a_{k+1} \otimes n = \sum_{j=1}^{k} \sum_{i=1}^{p} c_{k+1,i} b_j (x_{k+1,i} \otimes y_j) = x_{k+1,1} \otimes y_{k+1} ,$$

then

$$\sum_{j=1}^{k} \sum_{i=1}^{p} c_{k+1,i} \, b_j \, (x_{k+1,i} \otimes y_j) - x_{k+1,1} \otimes y_{k+1} = 0 .$$

By lemma 1.5, we know that $[\, x_{k+1,i} \otimes y_j \,]_{i,j}$ is a basis of $M_{k+1} \otimes N$.

then

$$\sum_{j=1}^{k} \sum_{i=1}^{p} c_{k+1,i} \, b_j \, (x_{k+1,i} \otimes y_j) - x_{k+1,1} \otimes y_{k+1} \neq 0 \text{ (contridiction).}$$

Therefore μ is not epimorphism. That is

$$(\prod_\alpha M_\alpha) \otimes N \not\cong \prod_\alpha (M_\alpha \otimes N) .$$

The following is to illustrate the mapping $(\prod_\alpha M_\alpha) \otimes_R N \to \prod_\alpha (M_\alpha \otimes_R N)$ which is not one to one but onto.

Example 2: Let p be a prime number, μ a homomorphism of $(\prod_{n=1}^{\infty} Z/p^nZ) \otimes Q \to \prod_{n=1}^{\infty} (Z/p^nZ \otimes Q)$. Then μ is epimorphism (since $Z/p^nZ \otimes Q = 0$). We show that μ is not monomorphism.

Since μ is epimorphism, we claim that $(\prod_{n=1}^{\infty} Z/p^nZ) \otimes Q \neq 0$. Let $S = Z - \{0\}$, $S^{-1} (\prod_{n=1}^{\infty} Z/p^nZ)$ be the localization of the Z-module $\prod_{n=1}^{\infty} Z/p^nZ$ with respect to S. By lemma 1.6, it is clear that

$$(\prod_{n=1}^{\infty} Z/p^nZ) \otimes Q \cong S^{-1} (\prod_{n=1}^{\infty} Z/p^nZ)$$

From the basic properties of localization, as element $x \in \prod_{n=1}^{\infty} Z/p^nZ$ becomes zero in $S^{-1} (\prod_{n=1}^{\infty} Z/p^nZ)$ if and only if there exists $s \in S$ such that $s \cdot x = 0$ in $\prod_{n=1}^{\infty} Z/p^nZ$. Thus $x = (x_1, x_2, \ldots) \in \prod_{n=1}^{\infty} Z/p^nZ$ is zero in $(\prod_{n=1}^{\infty} Z/p^nZ) \otimes Q$ if and only if $x_1 \ldots x_n \ldots$ are of bounded order.

Obviously, there are a lot of $x = (x_1, x_2 \ldots)$ so that $x_1, x_2, \ldots, x_n, \ldots$ are of unbounded order (e.g. $x = (1, 1, 1, \ldots)$). Hence $(\prod_{n=1}^{\infty} Z/p^nZ) \otimes Q \neq 0$.

In this section, we shall characterize modules which satisfy the distributive law of tensor product over direct product. We first give some lemmas which are used in the proof of the main theorem.

Lemma 3.1: Let M_i and N_i (i = 1, 2, 3, 4) be left (right) R-modules. Suppose that the diagram

$$\begin{array}{ccccccc} M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3 & \xrightarrow{f_3} & M_4 \\ \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & \downarrow \varphi_4 \\ N_1 & \xrightarrow{g_1} & N_2 & \xrightarrow{g_2} & N_3 & \xrightarrow{g_3} & N_4 \end{array}$$

is commutative, and that the rows are exact.

If φ_1 is an epimorphism and φ_2, φ_4 are monomorphism then φ_3 is monomorphism.

Proof: For any $x \in \ker \varphi_3$, $\varphi_3(x) = 0$, $g_3\varphi_3(x) = \varphi_4 f_3(x) = 0$. Since φ_4 is a monomorphism, $f_3(x) = 0$ implies $x \in \ker f_3 = \mathrm{Im} f_2$. There is $y \in M_2$ such that $f_2(y) = x$.

For $\varphi_2(y) \in N_2$, $g_2\varphi_2(y) = \varphi_3 f_2(y) = \varphi_3(x) = 0$, then $\varphi_2(y) \in \ker g_2 = \mathrm{Im}\, g_1$. There is $t \in N_1$ such that $g_1(t) = \varphi_2(y)$. Since φ_1 is an epimorphism, there is $s \in M_1$, such that $\varphi_1(s) = t$. Then $g_1\varphi_1(s) = g_1(t) = \varphi_2(y) = \varphi_2 f_1(s)$ implies $f_1(s) = y$ (since φ_2 is a monomorphism), $y \in \mathrm{Im}\, f_1 = \ker f_2$, hence $f_2(y) = 0 = x$. Thus $\ker \varphi_3 = 0$. Therefore φ_3 is a monomorphism.

Lemma 3.2: Let $[M_\alpha]_\alpha$ be a family of right R-modules, and A, B, C three left R-modules. If $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is an exact sequence then

$$\prod_{\alpha}(M_{\alpha}\otimes_R A) \to \prod_{\alpha}(M_{\alpha}\otimes_R B) \to \prod_{\alpha}(M_{\alpha}\otimes_R C) \to 0$$

is exact sequence.

Proof: By Lemma 1.1, we know that the sequence

$$M_{\alpha}\otimes_R A \xrightarrow{f_{\alpha}} M_{\alpha}\otimes_R B \xrightarrow{g_{\alpha}} M_{\alpha}\otimes_R C \to 0$$

is an exact sequence for each α. (where $f_{\alpha} = 1\otimes f$, $g_{\alpha} = 1\otimes g$)
Define

$$\overline{f} : \prod_{\alpha}(M_{\alpha}\otimes_R A) \to \prod_{\alpha}(M_{\alpha}\otimes_R B)$$

$$\overline{g} : \prod_{\alpha}(M_{\alpha}\otimes_R B) \to \prod_{\alpha}(M_{\alpha}\otimes_R C)$$

by

$$\overline{f}\,(m_{\alpha}\otimes a_{\alpha})_{\alpha} = (f_{\alpha}(m_{\alpha}\otimes a_{\alpha}))_{\alpha} = (m_{\alpha}\otimes f(a_{\alpha}))_{\alpha}$$

and

$$\overline{g}\,(m_{\alpha}\otimes b_{\alpha})_{\alpha} = (g_{\alpha}(m_{\alpha}\otimes b_{\alpha}))_{\alpha} = (m_{\alpha}\otimes g(b_{\alpha}))_{\alpha}.$$

Then $\overline{f}$ and $\overline{g}$ are R-module homomorphism. We show that Im $\overline{f}$ = ker $\overline{g}$. For each $(m_{\alpha}\otimes b_{\alpha})_{\alpha} \in \text{Im}\,\overline{f}$, there is $(m'_{\alpha}\otimes a_{\alpha})_{\alpha} \in \prod(M_{\alpha}\otimes_R A)$ such that $\overline{f}\,(m'_{\alpha}\otimes a_{\alpha})_{\alpha} = (m_{\alpha}\otimes b_{\alpha})_{\alpha} = (f_{\alpha}(m'_{\alpha}\otimes a_{\alpha}))_{\alpha} = (m'_{\alpha}\otimes f(a_{\alpha}))_{\alpha}$. Then $\overline{g}\,(m_{\alpha}\otimes b_{\alpha})_{\alpha} = \overline{g}\,(m'_{\alpha}\otimes f(a_{\alpha}))_{\alpha} = (g_{\alpha}(m_{\alpha}\otimes f(a_{\alpha})))_{\alpha} = (m'_{\alpha}\otimes g(f(a_{\alpha})))_{\alpha} = (m'_{\alpha}\otimes 0)_{\alpha} = 0$ implies Im $\overline{f} \subset$ ker $\overline{g}$. Conversely, we claim that ker $\overline{g} \subset$ Im $\overline{f}$. For each $(m_{\alpha}\otimes t_{\alpha})_{\alpha} \in$ ker $\overline{g}$, $\overline{g}\,(m_{\alpha}\otimes t_{\alpha})_{\alpha} = (m_{\alpha}\otimes g(t_{\alpha}))_{\alpha} = 0$ implies $g_{\alpha}(m_{\alpha}\otimes t_{\alpha}) = 0$. For each α, $m_{\alpha}\otimes t_{\alpha} \in \ker g_{\alpha} = \text{Im}\, f_{\alpha}$, there is $m'_{\alpha}\otimes a_{\alpha} \in M_{\alpha}\otimes_R A$ such that $f_{\alpha}(m'_{\alpha}\otimes a_{\alpha}) = m_{\alpha}\otimes t_{\alpha}$. Then $(f_{\alpha}(m'_{\alpha}\otimes a_{\alpha}))_{\alpha} = (m_{\alpha}\otimes t)_{\alpha} = \overline{f}\,(m'_{\alpha}\otimes a_{\alpha})_{\alpha}$ implies $(m_{\alpha}\otimes t_{\alpha})_{\alpha} \in$ Im $\overline{f}$. Thus Im $\overline{f}$ = ker $\overline{g}$.

Lemma 3.3: Let $[M_\alpha]_\alpha$ be a family of right R-modules, N a finitely generated free left R-module. Then

$$(\prod_\alpha M_\alpha) \otimes_R N \cong \prod_\alpha (M_\alpha \otimes_R N)$$

Proof: Since N is a finitely generated free left R-module, $N \cong \bigoplus_{i=1}^{n} R = R^n$ for some positive integer n. By lemma 1.1 (i), we know that

$$(\prod_\alpha M_\alpha) \otimes_R N \cong (\prod_\alpha M_\alpha) \otimes_R (\bigoplus_{i=1}^{n} R) \cong \bigoplus_{i=1}^{n} ((\prod M_\alpha) \otimes_R R) \cong \bigoplus_{i=1}^{n} (\prod M_\alpha).$$

$$\prod_\alpha (M_\alpha \otimes_R N) \cong \prod_\alpha (M_\alpha \otimes_R (\bigoplus_{i=1}^{n} R)) \cong \prod_\alpha [\bigoplus_{i=1}^{n} (M_\alpha \otimes_R R)] \cong \prod (\bigoplus_{i=1}^{n} M_\alpha).$$

Since $\bigoplus_{i=1}^{n} (\prod_\alpha M_\alpha) \cong \prod_\alpha (\bigoplus_{i=1}^{n} M_\alpha)$, $(\prod_\alpha M_\alpha) \otimes_R N \cong \prod_\alpha (M_\alpha \otimes_R N)$.

The following theorem is the main positive result of a sufficient condition for R-modules which will satisfy the distributive law of tensor product over direct product. We first give a definition which is used in the main theorem.

Definition: A right (left) module M over R is finitely presented, if there exist integers n and m such that

$$R^m \to R^n \to M \to 0$$

is exact.

Theorem: Let $[M_\alpha]_\alpha$ be a family of right R-modules, N a finitely presented left R-module. Then

$$(\prod_\alpha M_\alpha) \otimes_R N \cong \prod_\alpha (M_\alpha \otimes_R N)$$

Proof: Since N is a finitely presented R-module, there is an exact sequence $R^m \overset{f}{\to} R^n \overset{g}{\to} N \to 0$. By lemma 1.1 (ii) and lemma 3.2, the following sequences

$$(\prod_\alpha M_\alpha) \otimes_R R^m \overset{\overline{f_1}}{\to} (\prod_\alpha M_\alpha) \otimes_R R^n \overset{\overline{g_1}}{\to} (\prod_\alpha M_\alpha) \otimes_R N \to 0$$

and
$$\prod_\alpha (M_\alpha \otimes_R R^m) \overset{\overline{f_2}}{\to} \prod_\alpha (M_\alpha \otimes_R R^n) \overset{\overline{g_2}}{\to} \prod_\alpha (M_\alpha \otimes_R N) \to 0$$

are exact sequences.

By tensor product definition, we can define a module homomorphism $\mu : (\prod_\alpha M_\alpha) \otimes_R N \to \prod (M_\alpha \otimes_R N)$ by $\mu ((m_\alpha)_\alpha \otimes n) = (m_\alpha \otimes n)_\alpha$. By lemma 3.3, we have

$$(\prod_\alpha M_\alpha) \otimes_R R^m \cong \prod_\alpha (M_\alpha \otimes_R R^m)$$

and

$$(\prod_\alpha M_\alpha) \otimes_R R^n \cong \prod_\alpha (M_\alpha \otimes_R R^n) \ .$$

Consider the following diagram

$$\begin{array}{ccccccc}
(\prod_\alpha M_\alpha) \otimes_R R^m & \overset{\overline{f_1}}{\to} & (\prod_\alpha M_\alpha) \otimes_R R^n & \overset{\overline{g_1}}{\to} & (\prod_\alpha M_\alpha) \otimes_R N & \to & 0 \\
\downarrow i_1 & & \downarrow i_2 & & \downarrow \mu & & \\
\prod_\alpha (M_\alpha \otimes_R R_m) & \overset{\overline{f_2}}{\to} & \prod_\alpha (M_\alpha \otimes_R R^n) & \overset{\overline{g_2}}{\to} & \prod_\alpha (M_\alpha \otimes_R N) & \to & 0
\end{array}$$

where i_1, i_2 are isomorphisms and $\overline{g_1}$, $\overline{g_2}$ are epimorphisms.

We claim that the diagram is commutative. For every $(m_\alpha)_\alpha \otimes \gamma \in (\prod_\alpha M_\alpha) \otimes_R R^m$, $i_2 \overline{f_1} ((m_\alpha)_\alpha \otimes \gamma) = i_2 ((m_\alpha)_\alpha \otimes f(\gamma)) = (m_\alpha \otimes f(\gamma))_\alpha$, $\overline{f_2} i_1 ((m_\alpha)_\alpha \otimes \gamma) = \overline{f_2} (m_\alpha \otimes \gamma)_\alpha = (m_\alpha \otimes f(\gamma))_\alpha$. Thus $i_2 \overline{f_1} = \overline{f_2} i_1$. For every $(m_\alpha)_\alpha \otimes n \in (\prod_\alpha M_\alpha) \otimes_R R^n$. $\mu \overline{g_1} ((m_\alpha)_\alpha \otimes n) = \mu ((m_\alpha)_\alpha \otimes g(n)) = (m_\alpha \otimes g(n))_\alpha$, $\overline{g_2} i_2 ((m_\alpha)_\alpha \otimes n) = \overline{g_2} ((m_\alpha \otimes \gamma)_\alpha) = (m_\alpha \otimes g(\gamma))_\alpha$. Thus $\overline{g_2} i_2 = \mu \overline{g_1}$.

By lemma 3.1, we know that μ is monomorphism. We show that μ is epimorphism. For every $z \in \prod_{\alpha} (M_\alpha \otimes_R N)$, there is $y \in \prod_{\alpha} (M_\alpha \otimes_R R^n)$ such that $\overline{g}_2 (y) = z$. Since i_2 is isomorphism, there is $x \in (\prod M_\alpha) \otimes_R R^n$ such that $i_2 (x) = y$. Then $\overline{g}_2 i_2 (x) = \overline{g}_2 (y) = z = \mu \overline{g}_1 (x)$. Since $\overline{g}_1 (x) \in (\prod_{\alpha} M_\alpha) \otimes_R N$, μ is epimorphism. Therefore μ is isomorphism, that is,

$$(\prod_{\alpha} M_\alpha) \otimes_R N \cong \prod_{\alpha} (M_\alpha \otimes_R N)$$

Remark: When R is a Noetherian ring, a module is finitely presented if and only if it is finitely generated.

Corollary: Let R be a Noetherian ring, $[M_\alpha]_\alpha$ a family of right R-modules and N a finitely generated left R-module. Then $(\prod_{\alpha} M_\alpha) \otimes_R N \cong \prod_{\alpha} (M_\alpha \otimes_R N)$.

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