# Stability Analysis for Fuzzy Control of Passive Tuned Mass Damper Subjected to External Force

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## Abstract

This paper investigates the effectiveness of a passive Tuned Mass Damper (TMD) and active fuzzy controllers in reducing the structural responses under the external force. In general, TMD is good for linear system. We proposed here a fuzzy controller to deal with the nonlinear system. A robustness design of fuzzy control via model-based approach is proposed in this paper to overcome the effect of modeling error between nonlinear systems and Takagi-Sugeno (T-S) fuzzy models. A stability criterion in terms of Lyapunov's direct method is derived to guarantee the stability of nonlinear interconnected TMD systems. Based on the decentralized control scheme and this criterion, a set of model-based fuzzy controllers is then synthesized via the technique of parallel distributed compensation (PDC) to stabilize the nonlinear interconnected TMD system. Finally, an example is given to illustrate the concepts discussed throughout this paper.

Key words：T-S fuzzy models, fuzzy control, Lyapunov theroy.

# Passive Tuned Mass Damper 受外力作用下模糊控制之穩定性分析

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## 摘　　要

本論文研究被動調和質量阻尼系統受外力作用下，結合主動模糊控制在降低結構反應之有效性。模糊控制經由本文所提 model-based approach，以克服非線性系統由 Tagagi-Sugeno (T-S)系統模式模擬的模式誤差。TMD 系統穩定的準則由 Lyapunov 理論推得。此穩定準則與 decentralized control scheme 並透過 parallel distributed compensation (PDC) 技術，可確保非線性 TMD 系統具有穩定性。最後將以實例驗證本論文所提之控制演算法。

**關鍵字：** T-S 模糊模式，模糊控制，Lyapunov 理論

# 1. Introduction

Traditional structural design depends on structural strength and capability to dissipate energy to resist dynamic forces such as machine loading, wind forces and earthquakes. The use of passive tuned mass dampers (TMDs) as a means to control and reduce the vibration of dynamic systems was first proposed by Frahm in 1909 [1]. Since then, much research has been done to investigate the control effectiveness of passive TMDs (see [2-3] and the references therein). These articles show that the TMDs are suitable for a linear resonant system and it will be useful only for the frequency of TMD close to the primary structure. Nevertheless, for a relatively small displacement, the restoring force of the spring can be modeled linearly. Nonlinear stiffness is considered for a large displacement so that the TMD is not appropriate [4]. The objective of this paper is to derive a stability criterion for model-based fuzzy controller to guarantee the uniformly ultimately bounded (UUB) stable of nonlinear interconnected systems.

Many control methods have been proposed to overcome the difficulty of dealing with nonlinear systems. Since the design of control strategy of nonlinear systems is a difficult process and the plants are in general nonlinear in practical sense. Due to the complexity of designing a general control scheme for a nonlinear system, we proposed here a simplified model. In the past few years, fuzzy-rule-based modeling has become an active research field because of its unique merits in solving complex nonlinear system identification and control problems. In attempt to attain more flexibility and more effective capability of handling and processing uncertainties in complicated and ill-defined systems, Zadeh [5] proposed a linguistic approach as the model of human thinking, which introduced the fuzziness into systems theory [6]. Unlike traditional modeling, fuzzy rule-based modeling is essentially a multimodel approach in which individual rules are combined to describe the global behavior of the system [7].

Fuzzy control has attracted a great deal of attention from both the academic and industrial communities in the past few years, and there have been many successful applications. In spite of the success, there are still many basic issues that remain to be further addressed. Stability analysis and systematic design are certainly among the most important issues for fuzzy control systems. Recently, there have been significant research efforts on these issues [8-13]. All of them, however, neglect the modeling error between nonlinear systems and fuzzy models. Existence of the modeling error may be a potential source of instability for control designs that have been based on the assumption that the fuzzy model exactly matches the plant.

In this paper, a stability criterion in terms of Lyapunov's direct method is derived in this study to guarantee the stability of nonlinear TMD systems. According to this criterion and the control scheme, a model-based fuzzy controller is then synthesized to stabilize the nonlinear TMD system. Moreover, the system is represented by a Takagi-Sugeno (T-S) type fuzzy model. In this type of fuzzy model, each fuzzy implication is expressed by a linear system model, which allows us to use linear feedback control as in the case of feedback stabilization. The control design is carried out based on the fuzzy model via the parallel distributed compensation (PDC) scheme. The idea is that a linear feedback control is designed for each local linear model. The resulting overall fuzzy controller, which is nonlinear in general, is a fuzzy blending of each individual linear controller [8, 11].

This paper is organized as follows. First, the T-S fuzzy model is briefly reviewed and the system description is presented. Then, a stability criterion is derived to guarantee the stability of nonlinear interconnected systems. Next, a TMD is used to reduce the vibration of dynamic linear system but it fails for the nonlinear system. So, a set of model-based fuzzy controllers via the technique of PDC is proposed to overcome the influence of modeling error and stabilize the nonlinear interconnected TMD system. Finally, a numerical example of nonlinear interconnected TMD system with simulations is given to illustrate the results, and the conclusions are drawn.

# 2. System Description

Consider a nonlinear interconnected system $\boldsymbol{N}$ composed of $J$ subsystems $N_j$, $j=1,2,\cdots,J$. The $j$th subsystem $N_j$ is described as follows:

$$\dot{x}_j(t)=f_j(x_j(t),u_j(t))+\sum_{\substack{n=1\\n\neq j}}^{J}C_{nj}x_n(t)+\phi_j(t) \tag{2.1}$$

where $f_j$ is the nonlinear vector-valued function, $x_j(t)$ is the state vector, $u_j(t)$ is the input vector, $\phi_j(t)$ denotes the external force and $C_{nj}$ is the interconnection matrix between the $n$th subsystem and $j$th subsystems.

**Definition 2.1** [4]**:** The solution of a dynamic system are said to be uniformly ultimately bounded (UUB) if there exist positive constants $\beta$ and $\kappa$, and for every $\delta\in(0,\kappa)$ there is a positive constant $T=T(\delta)$, such that

$$\|x(t_0)\|<\delta\Rightarrow\|x(t)\|\leq\beta,\forall t\geq t_0+T$$

**Assumption 2.1:** $f_j(0)=0$, $j=1,2,\cdots,J$ so that the origin is an equilibrium point of the system (2.1).

In a little more than a decade ago, a fuzzy dynamical model had been developed primarily from the pioneering work of Takagi and Sugeno [14] to represent local linear input/output relations of nonlinear systems. This dynamical model is described by fuzzy IF-THEN rules and it will be employed here to handle the control design problem of the nonlinear interconnected system $\boldsymbol{N}$. The $i$th rule of this fuzzy model for the nonlinear interconnected subsystem $N_j$ is proposed as the following form:

Rule $i$: IF $x_{1j}(t)$ is $M_{i1j}$ and $\cdots$ and $x_{gj}(t)$ is $M_{igj}$

$$\text{THEN}\quad \dot{x}_j(t)=A_{ij}x_j(t)+B_{ij}u_j(t)+\phi_j(t) \tag{2.2}$$

where $x_j^T(t)=[x_{1j}(t),x_{2j}(t),\cdots,x_{gj}(t)]\in R^{1\times g}$ denotes the state vector, $x_{kj}$ is the kth state component, k=1,…g.

$u_j^T(t)=[u_{1j}(t),\ u_{2j}(t),\cdots,u_{mj}(t)]\in R^{1\times m}$ denotes the control input, $u_{pj}$ is the kth state component, p=1,…m.

$\phi_j^T(t)=[\phi_{1j}(t),\ \phi_{2j}(t),\cdots,\phi_{zj}(t)]\in R^{1\times z}$ denotes the unknown disturbances with a known upper bound $\phi_{upj}(t)\geq\|\phi_j(t)\|$, $\phi_{upj}(t)$ is the upper bound for the nonlinear interconnected subsystem $N_j$. $i=1,2,\cdots,r_j$ and $r_j$ is the number of IF-THEN rules; $A_{ij}$, and $B_{ij}$ are constant matrices with appropriate dimensions; $M_{ipj}$ ($p=1,2,\cdots,g$) are the fuzzy sets, and $x_{1j}(t)\sim x_{gj}(t)$ are the premise variables. The final state of this fuzzy dynamic model is inferred as follows:

$$\dot{x}_j(t)=\frac{\sum_{i=1}^{r_j} w_{ij}(t)[A_{ij}x_j(t)+B_{ij}u_j(t)+\phi_j(t)]}{\sum_{i=1}^{r_j} w_{ij}(t)}$$

$$=\sum_{i=1}^{r_j} h_{ij}(t)(A_{ij}x_j(t)+B_{ij}u_j(t))+\phi_j(t) \tag{2.3}$$

with

$$w_{ij}(t)\equiv\prod_{p=1}^{g} M_{ipj}(x_{pj}(t)), \quad h_{ij}(t)\equiv\frac{w_{ij}(t)}{\sum_{i=1}^{r_j} w_{ij}(t)} \tag{2.4}$$

in which $M_{ipj}(x_{pj}(t))$ is the grade of membership of $x_{pj}(t)$ in $M_{ipj}$. In this paper, it is assumed that $w_{ij}(t)\ge 0$, $i=1,2,\cdots,r_j$; $j=1,2,\cdots,J$ and $\sum_{i=1}^{r_j} w_{ij}(t)>0$ for all $t$. Therefore, $h_{ij}(t)\ge 0$ and $\sum_{i=1}^{r_j} h_{ij}(t)=1$ for all $t$.

In the next section, the concept of PDC scheme is utilized to design fuzzy controllers.

# 3. Parallel Distributed Compensation

According to the decentralized control scheme, a set of model-based fuzzy controllers is synthesized via the technique of parallel distributed compensation (PDC) to stabilize the nonlinear interconnected system ***N***. The concept of PDC scheme is that each control rule is distributively designed for the corresponding rule of a T-S fuzzy model. The fuzzy controller shares the same fuzzy sets with the fuzzy model in the premise parts [9]. Since each rule of the fuzzy model is described by a linear state equation, a linear control theory can be used to design the consequent parts of a fuzzy controller. The resulting overall fuzzy controller, nonlinear in general, is achieved by fuzzy blending of each individual linear controller.

Hence, the *j*th model-based fuzzy controller can be described as follows:

Rule *i*: IF $x_{1j}(t)$ is $M_{i1j}$ and $\cdots$ and $x_{gj}(t)$ is $M_{igj}$

THEN $u_j(t)=-K_{ij}x_j(t)$, (3.1)

where $i=1, 2,\cdots, r_j$. The final output of this fuzzy controller is

$$u_j(t)=-\frac{\sum_{i=1}^{r_j} w_{ij}(t)K_{ij}x_j(t)}{\sum_{i=1}^{r_j} w_{ij}(t)}=-\sum_{i=1}^{r_j} h_{ij}(t)K_{ij}x_j(t). \tag{3.2}$$

# 4. Robustness Design of Fuzzy Control

In this section, the stability of the nonlinear interconnected system $N$ is examined under the influence of modeling error. In subsection 4.1, the issue of modeling error is addressed and the guarantee of stability of $N$ is given in subsection 4.2.

## 4.1 Modeling Error

Substituting Eq. (3.2) into Eq. (2.1) yields the *j*th ( $j=1,2,\cdots,J$ ) closed-loop nonlinear subsystem $\overline{N}_j$ as follows:

$$\dot{x}_j(t) = \sum_{i=1}^{r_j}\sum_{l=1}^{r_j} h_{ij}(t)h_{lj}(t)[(A_{ij}-B_{ij}K_{lj})x_j(t)] + \phi_j(t) + \overline{f_j}(x_j(t)) + \sum_{\substack{n=1\\n\neq j}}^{J} C_{nj}x_n(t)$$

$$-\sum_{i=1}^{r_j}\sum_{l=1}^{r_j} h_{ij}(t)h_{lj}(t)[(A_{ij}-B_{ij}K_{lj})x_j(t)]$$

$$= \sum_{i=1}^{r_j}\sum_{l=1}^{r_j} h_{ij}(t)h_{lj}(t)[(A_{ij}-B_{ij}K_{lj})x_j(t)] + \phi_j(t) + e_j(t) + \sum_{\substack{n=1\\n\neq j}}^{J} C_{nj}x_n(t) \tag{4.1}$$

where $\overline{f_j}(x_j(t)) \equiv f_j(x_j(t),u_j(t))$ with $u_j(t) = -\sum_{i=1}^{r_j} h_{ij}(t)K_{ij}x_j(t)$

$$e_j(t) \equiv [\overline{f_j}(x_j(t)) - \sum_{i=1}^{r_j}\sum_{l=1}^{r_j} h_{ij}(t)h_{lj}(t)(A_{ij}-B_{ij}K_{lj})x_j(t)], \tag{4.2}$$

and $e_j(t)$ denotes the modeling error between the *j*th closed-loop nonlinear subsystem (4.1) and the close-loop fuzzy model ((2.3)+(3.2)).

Suppose that there exists a bounding matrix $\Delta H_{ilj}$ such that

$$\| e_j(t) \| \le \left\| \sum_{i=1}^{r_j}\sum_{l=1}^{r_j} h_{ij}(t)h_{lj}(t)\Delta H_{ilj}x_j(t) \right\| \tag{4.3}$$

for the trajectory $x_j(t)$ and the bounding matrix $\Delta H_{ilj}$ can be described as follows [15, 16]:

$$\Delta H_{ilj} = \delta_{ilj}\overline{H}_j \tag{4.4}$$

where $\| \delta_{ilj} \| \le 1$, for $i,\ l=1,2,\cdots,r_j$, $j=1,2,\cdots,J$. From Eqs. (4.3–4.4), we have

$$e_j^T(t)\,e_j(t) \le \{\sum_{i=1}^{r_j}\sum_{l=1}^{r_j} h_{ij}(t)h_{lj}(t)\Delta H_{ilj}x_j(t)\}^T \{\sum_{i=1}^{r_j}\sum_{l=1}^{r_j} h_{ij}(t)h_{lj}(t)\Delta H_{ilj}x_j(t)\}$$

$$= \{\sum_{i=1}^{r_j}\sum_{l=1}^{r_j} h_{ij}(t)h_{lj}(t)\delta_{ilj}\overline{H}_j x_j(t)\}^T \{\sum_{i=1}^{r_j}\sum_{l=1}^{r_j} h_{ij}(t)h_{lj}(t)\delta_{ilj}\overline{H}_j x_j(t)\}$$

$$\leq \left\| \sum_{i=1}^{r_j} \sum_{l=1}^{r_j} h_{ij}(t) h_{lj}(t) \delta_{ilj}^T \right\| \left\| \sum_{i=1}^{r_j} \sum_{l=1}^{r_j} h_{ij}(t) h_{lj}(t) \delta_{ilj} \right\| [\overline{H}_j x_j(t)]^T [\overline{H}_j x_j(t)]$$

$$\leq [\overline{H}_j x_j(t)]^T [\overline{H}_j x_j(t)] \,. \tag{4.5}$$

That is to say the modeling error in Eq. (4.2) can be bounded by the specified structured bounding matrix $\overline{H}_j$.

**Remark 4.1:** The procedures for determining $\delta_{ilj}$ and $\overline{H}_j$ are described by the following simple example. Assuming that the possible bounds for all elements in $\Delta H_{ilj}$ are

$$\Delta H_{ilj} = \begin{bmatrix} \Delta h_{ilj}^{11} & \Delta h_{ilj}^{12} \\ \Delta h_{ilj}^{21} & \Delta h_{ilj}^{22} \end{bmatrix}$$

where $-\varepsilon_j^{rs} \leq \Delta h_{ilj}^{rs} \leq \varepsilon_j^{rs}$ for some $\varepsilon_j^{rs}$, $r, s = 1, 2$ and $j = 1, 2, \cdots, J$.

One possible description for the bounding matrix $\Delta H_{ilj}$ is

$$\Delta H_{ilj} = \begin{bmatrix} \delta_{ilj}^{11} & 0 \\ 0 & \delta_{ilj}^{22} \end{bmatrix} \begin{bmatrix} \varepsilon_j^{11} & \varepsilon_j^{12} \\ \varepsilon_j^{21} & \varepsilon_j^{22} \end{bmatrix} = \delta_{ilj} \overline{H}_j$$

where $-1 \leq \delta_{ilj}^{rr} \leq 1$ for $r = 1, 2$. It is noticed that $\delta_{ilj}$ can be chosen by other forms as long as $\| \delta_{ilj} \| \leq 1$. Then, we check the validity of Eq. (4.3) in the simulation. If it is not satisfied, we can expand the bounds for all elements in $\Delta H_{ilj}$ and repeat the design procedures until Eq. (4.3) holds.

## 4.2Stability in the Presence of Modeling Error

In the following, a stability criterion is proposed to guarantee the stability of the closed-loop nonlinear interconnected system $\overline{N}$ which consists of *J* closed-loop subsystems described in Eq. (4.1). Prior to examination of stability of $\overline{N}$, an useful concept is given below.

**Lemma 4.1** [17]**:** For real matrices *A* and *B* with appropriate dimensions, we have

$$A^T B + B^T A \leq \sigma A^T A + \sigma^{-1} B^T B \qquad \text{where } \sigma \text{ is a positive constant.}$$

**Theorem 4.1:** The closed-loop nonlinear interconnected system $\overline{N}$ is stable, if there exist symmetric positive definite matrices $P_j$ and positive constants $\beta_j$, $\lambda$, $\gamma$ and the feedback gains $K_{ij}$'s shown in Eq. (3.2) are chosen to satisfy

$$\hat{\lambda}_{inj} = \lambda_M(Q_{inj}) < 0 \qquad \text{for } i = 1, 2, \cdots, r_j; \qquad n, j = 1, 2, \cdots, J \tag{4.6}$$

where

$$Q_{inj} = \{ \frac{1}{J} [(A_{ij} - B_{ij} K_{ij})^T P_j + P_j (A_{ij} - B_{ij} K_{ij})] + [\lambda (\frac{J-1}{J}) I + \lambda^{-1} P_j C_{nj} C_{nj}^T P_j ]$$

$$+ \frac{1}{J} [\beta_j \overline{H}_j^T \overline{H}_j + \beta_j^{-1} P_j^2 ] + \frac{1}{J} \gamma_j^{-1} P_j^2 \} , \tag{4.7}$$

Moreover, $\lambda_M(Q_{inj})$ denotes the maximum eigenvalue of $Q_{inj}$.

The proof is in the appendix.

# 5. Algorithm

Based on the above analysis, the complete design procedure can be summarized in the following algorithm.

**Problem:** For a given nonlinear interconnected system ***N***, how do we synthesize a set of decentralized fuzzy controllers to stabilize ***N*** ?

The problem described above can be solved in the following steps.

Step 1: Select the fuzzy plant rules and membership functions for each nonlinear subsystem $N_j$ to establish its fuzzy model.

Step 2: Synthesize a set of decentralized model-based fuzzy controllers via the concept of PDC scheme.

Step 3: Based on Remark 4.1, the bounding matrix $\Delta H_{ilj} (= \delta_{ilj} \overline{H}_j)$, for $i, l = 1, 2, \cdots, r_j$, $j = 1, 2, \cdots, J$, are chosen to satisfy Eq. (4.3).

Step 4: If there exist some positive definite matrices $P_j$ and the feedback gains $K_{ij}$'s to satisfy the stability conditions of Theorem 4.1 via LMI (linear matrix inequality) optimization algorithms, the nonlinear interconnected system ***N*** can be stabilized by the synthesized fuzzy controllers in Step 2. Otherwise, repeat Steps 2-3 to find appropriate fuzzy controllers and the bounding matrix $\Delta H_{ilj}$ $(= \delta_{ilj} \overline{H}_j)$ such that the stability criterion is satisfied.

# 6. Example

The objective of this section is to synthesize a set of T-S fuzzy controllers such that the nonlinear interconnected TMD system $N$ which is composed of two subsystems can be stabilized.

**Subsystem 1:**

$$\begin{cases} \dot{x}_{11}(t) = 10x_{21}(t) \\ \dot{x}_{21}(t) = -0.1681\,x_{11}(t) + 1.6641\times10^{-7}\,x_{11}^3(t) - 2.531\times10^{-3}\,x_{21}(t) + 1.6641\times10^{-3}\,x_{12}(t) \\ \qquad\qquad + 2.506\times10^{-5}\,x_{22}(t) + \cos(1.29\,t) + 5\,u_1(t) \end{cases} \tag{6.1}$$

**Subsystem 2:**

$$\begin{cases} \dot{x}_{12}(t) = 10\,x_{22}(t) \\ \dot{x}_{22}(t) = -0.16641\,x_{12}(t) + 1.6641\times10^{-7}\,x_{12}^3(t) - 2.506\times10^{-3}\,x_{22}(t) + 0.16641\,x_{11}(t) \\ \qquad\qquad - 1.664\times10^{-7}\,x_{11}(t)x_{12}^2(t) + 2.506\times10^{-3}\,x_{21}(t) + 4.5\,u_2(t) \end{cases} \tag{6.2}$$

Where $\varpi$ is frequency of external force; $\omega_1$ and $\omega_2$ denote nature frequency of primary structure and nature frequency of TMD. $\xi$ $(=\overline{\omega}/\omega)$ denotes frequency ratio.

From Eq. (6.1) and Eq. (6.2), $\omega_1$ and $\omega_2$ can be known as 1.29. Fig. 6.1 shows the effectiveness of a linear TMD system in reducing the response due to an external force with $\overline{\omega}=1.29$ and initial conditions $x_{11}(0)=x_{21}(0)=x_{12}(0)=x_{22}(0)=0$ when nonlinear terms are assumed as zero. Similarly, Fig. 6.2 shows the dynamic magnification factor where restoring force is a linear function. So, the passive TMD is appropriate when the frequency of external excitation is close to the structure. But, the restore force of spring stiffness is nonlinear in actual systems. It is no use for TMD system shown in Fig. 6.3 with nonlinear stiffness shown in Eqs. (6.1-6.2) [4, 18].

Moreover, the interconnection matrices among two nonlinear subsystems are obtained as follows:

$$C_{21}=10^{-5}\times\begin{bmatrix}0 & 0\\ 166.41 & 2.506\end{bmatrix},\qquad C_{12}=10^{-3}\times\begin{bmatrix}0 & 0\\ 16641 & 2.506\end{bmatrix}. \tag{6.3}$$

How do we synthesize two T-S fuzzy controllers to stabilize the system $\boldsymbol{N}$ ?

**Solution:** We can solve this problem according to the following steps.

**Step 1:** Establish a T-S fuzzy model for each nonlinear interconnected subsystem. To minimize the design effort and complexity, we try to use as few rules as possible. Hence, the subsystems $(6.1-6.2)$ are approximated with the following fuzzy models:

**T-S fuzzy model of subsystem 1:**

Rule 1: IF $x_{11}(t)$ is $M_{111}$

THEN $\dot{x}_1(t)=A_{11}x_1(t)+B_{11}u_1(t)$,

Rule 2: IF $x_{11}(t)$ is $M_{211}$

THEN $\dot{x}_1(t)=A_{21}x_1(t)+B_{21}u_1(t)$

where

$$x_1^T(t)=[x_{11}(t)\ \ x_{21}(t)]\ ,\ A_{11}=\begin{bmatrix}0 & 10\\ -0.1681 & -0.0025\end{bmatrix}\ ,\ A_{21}=\begin{bmatrix}0 & 10\\ -0.1680 & -0.0025\end{bmatrix}\ ,\ B_{11}=\begin{bmatrix}0\\ 5\end{bmatrix}\ ,$$
$$B_{21}=\begin{bmatrix}0\\ 5\end{bmatrix} \tag{6.4}$$

and the membership functions for Rule 1 and Rule 2 are

$$M_{111}(x_{11}(t))=\frac{1}{\left[1+\left|\frac{1-x_{11}(t)}{2}\right|\right]^2},\qquad M_{211}(x_{11}(t))=1-M_{111}(x_{11}(t)).$$

**T-S fuzzy model of subsystem 2:**

Rule 1: IF $x_{12}(t)$ is $M_{112}$

THEN $\dot{x}_2(t)=A_{12}x_2(t)+B_{12}u_2(t)$,

Rule 2: IF $x_{12}(t)$ is $M_{212}$

THEN $\dot{x}_2(t)=A_{22}x_2(t)+B_{22}u_2(t)$

where

$$x_2^T(t)=[x_{12}(t)\ \ x_{22}(t)]\ ,\ A_{12}=\begin{bmatrix}0 & 10\\ -0.1664 & -0.0025\end{bmatrix}\ ,\ A_{22}=\begin{bmatrix}0 & 10\\ -0.1663 & -0.0025\end{bmatrix}\ ,\ B_{12}=\begin{bmatrix}0\\ 4.5\end{bmatrix}\ ,$$
$$B_{22}=\begin{bmatrix}0\\ 4.5\end{bmatrix} \tag{6.5}$$

and membership functions for Rule 1 and Rule 2 are

$$\begin{cases} M_{112}(x_{12}(t)) = \dfrac{2}{3\pi}x_{12}(t)+1 & \text{when } -\dfrac{3\pi}{2} \le x_{12}(t) \le 0 \\ M_{112}(x_{12}(t)) = -\dfrac{2}{3\pi}x_{12}(t)+1 & \text{when } 0 < x_{12}(t) \le \dfrac{3\pi}{2} \\ M_{112}(x_{12}(t)) = 0 & \text{otherwise,} \end{cases}$$

$$M_{212}(x_{12}(t)) = 1 - M_{112}(x_{12}(t)).$$

**Step 2:** In order to stabilize the nonlinear interconnected system $N$, two model-based fuzzy controllers designed via the concept of PDC scheme are synthesized as follows.

**Fuzzy controller of subsystem 1:**

Rule 1: IF $x_{11}(t)$ is $M_{111}$ THEN $u_1(t) = -K_{11}x_1(t)$,

Rule 2: IF $x_{11}(t)$ is $M_{211}$ THEN $u_1(t) = -K_{21}x_1(t)$. (6.6)

**Fuzzy controller of subsystem 2:**

Rule 1: IF $x_{12}(t)$ is $M_{112}$ THEN $u_2(t) = -K_{12}x_2(t)$,

Rule 2: IF $x_{12}(t)$ is $M_{212}$ THEN $u_2(t) = -K_{22}x_2(t)$. (6.7)

**Step 3:** In accordance with Remark 4.1, two specified structured bounding matrices are chosen as

$$\overline{H}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 2.5\times10^{-7} \end{bmatrix}, \quad \overline{H}_2 = \begin{bmatrix} 0.001 & 0.001 \\ 0.001 & 0.6 \end{bmatrix}, \quad \delta_{ilj} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{for } i, l = 1, 2; \; j = 1, 2. \tag{6.8}$$

**Step 4:** To meet the stability condition of Theorem 4.1, the matrices $Q_{inj}$'s in Eq. (4.7) are chosen to be negative definite. Hence, based on Eqs. (6.4–6.8), we can obtain the following positive definite matrices $P_j$ ($j$ =1, 2) and $K_{ij}$'s via LMI optimization algorithms such that the matrices $Q_{inj}$'s are negative definite with $\beta_1 = \beta_2 = 0.01$, $\lambda = 0.1$ and $\gamma = 0.1$:

$$P_1 = \begin{bmatrix} 0.1233 & 0.0461 \\ 0.0461 & 0.0427 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 0.0705 & 0.0255 \\ 0.0255 & 0.0387 \end{bmatrix}, \tag{6.9}$$

$$K_{11} = [11.9664 \quad 9.9995], \qquad K_{21} = [7.4664 \quad 7.9995],$$

$$K_{12} = [6.6297 \quad 7.7772], \qquad K_{22} = [6.6297 \quad 7.7772] \tag{6.10}$$

Furthermore, the assumption of (4.3) for $j = 1, 2$ are satisfied from the illustration in Figs. 6.4 – 6.5 with initial conditions, $x_{11}(0) = 1$, $x_{21}(0) = -1$, $x_{12}(0) = 0.1$ and $x_{22}(0) = -0.1$. Substituting Eqs. (6.4 – 6.10) into Eq. (4.7) yields that all the matrices $Q_{inj}$'s ($j = 1, 2$) are negative definite. Therefore, based on Theorem 4.1, the T-S fuzzy controllers described in Eqs. (6.6 – 6.7) can stabilize the nonlinear interconnected TMD system $N$. Simulation results of each closed-loop subsystem $\overline{N}_j$ ($j = 1, 2$) are illustrated in Figs. 6.6 – 6.7.

# 7. Conclusions

In order to ensure the stability of nonlinear interconnected TMD systems, a stability criterion is derived from Lyapunov's direct method. According to this criterion and the decentralized control scheme, a set of model-based fuzzy controllers is synthesized to stabilize the nonlinear interconnected TMD system and overcome the influence of modeling error. Similarly, the common P matrix of the criterion is obtained by using linear matrix inequality (LMI) optimization algorithms to solve the robust fuzzy control problem. So, the proposed fuzzy control can be applied to any robust control design of nonlinear interconnected systems. Finally, a numerical example with simulations is provided to demonstrate the results

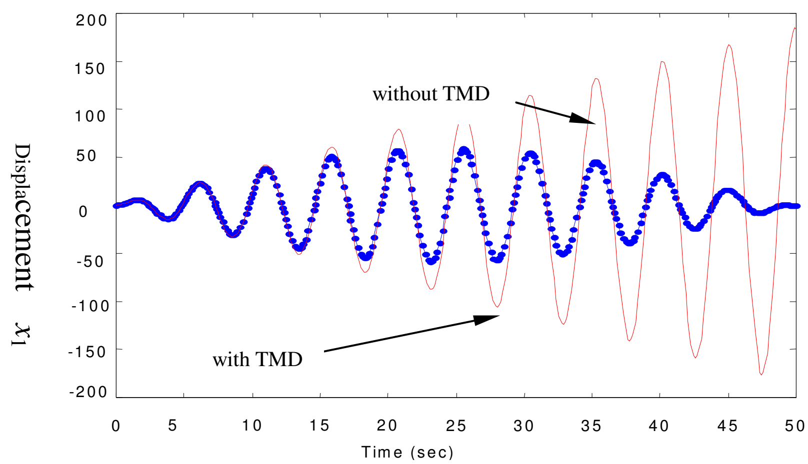


Fig. 6.1. The effectiveness of a TMD system.

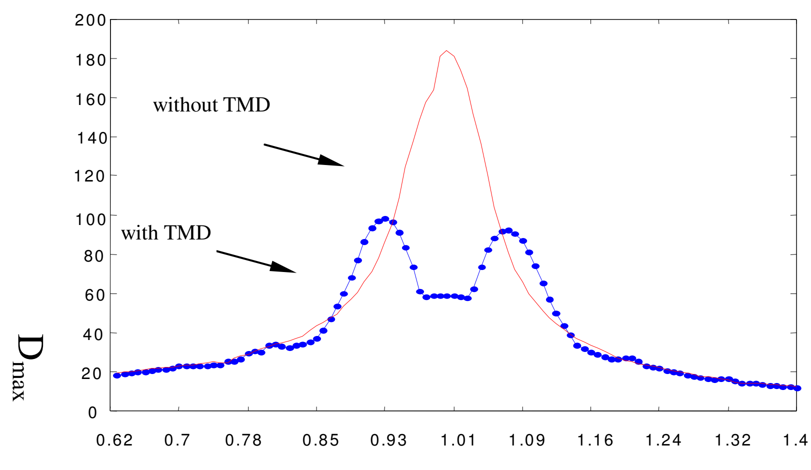


Fig. 6.2. The effectiveness of a TMD system with linear stiffness *k*(x).

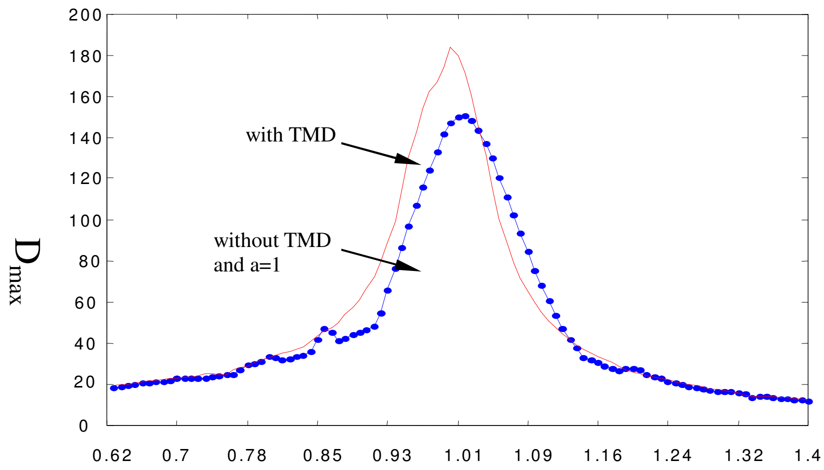


Fig. 6.3. Dynamic magnification factor of a TMD system with nonlinear stiffness $k$(x).

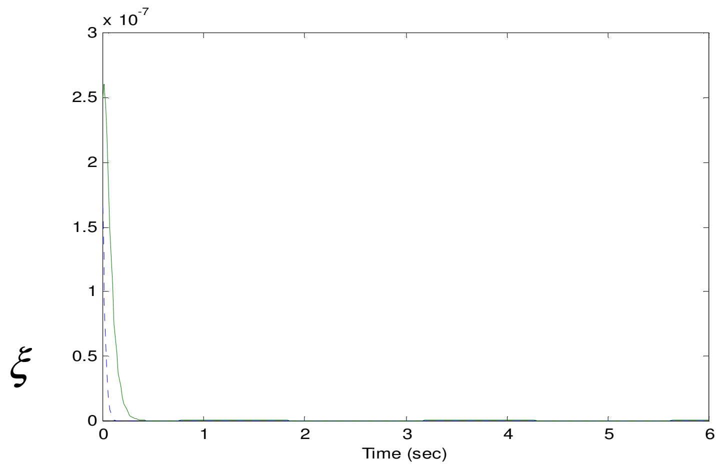


Fig. 6.4. The plots of $\left\| \overline{f_1}(x_1(t)) - \sum_{i=1}^{2}\sum_{l=1}^{2} h_{i1}(t)h_{l1}(t)(A_{i1} - B_{i1}K_{l1})x_1(t) \right\|$ (dashed line) and $\left\| \sum_{i=1}^{2}\sum_{l=1}^{2} h_{i1}(t)h_{l1}(t)\Delta H_{il1}x_1(t) \right\|$ (solid line).

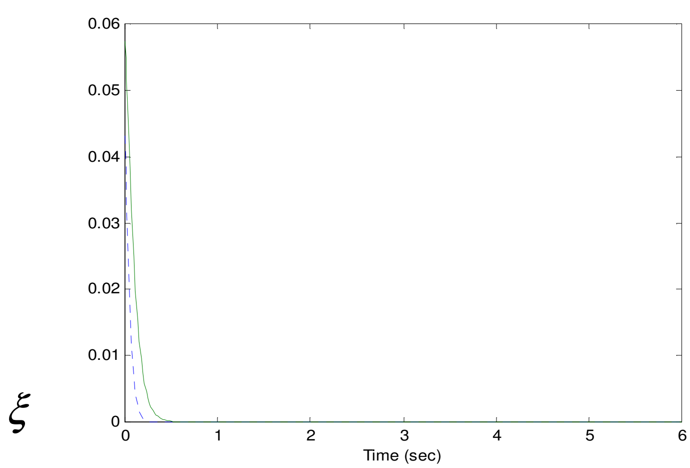


Fig. 6.5. The plots of $\left\| \overline{f_2}(x_2(t)) - \sum_{i=1}^{2}\sum_{l=1}^{2} h_{i2}(t)h_{l2}(t)(A_{i2} - B_{i2}K_{l2})x_2(t) \right\|$ (dashed line) and $\left\| \sum_{i=1}^{2}\sum_{l=1}^{2} h_{i2}(t)h_{l2}(t)\Delta H_{il2}x_2(t) \right\|$ (solid line).

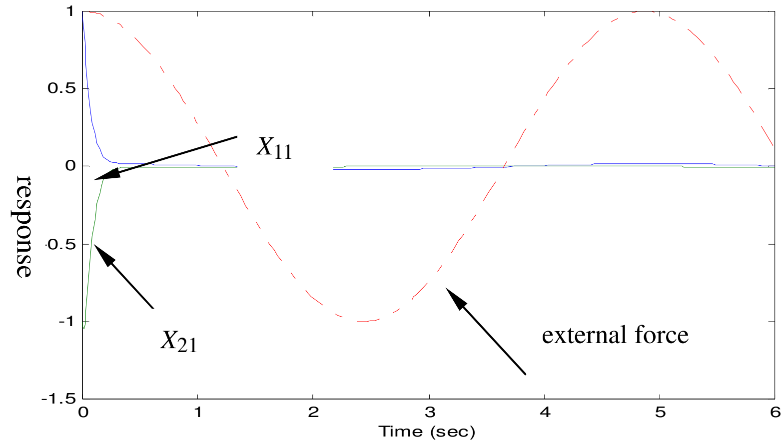


Fig. 6.6. The state response of system 1.

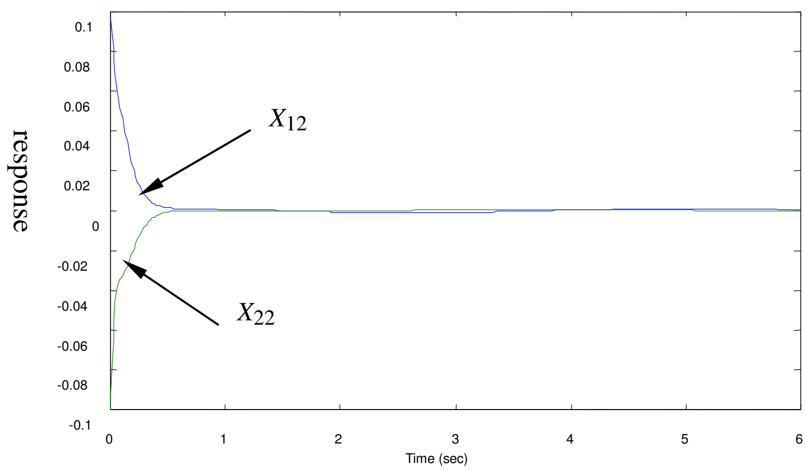


Fig. 6.7. The state response of system 2.

# V. Appendix: Proof of Theorem 4.1

Let the Lyapunov function for the closed-loop nonlinear interconnected system $\overline{\boldsymbol{N}}$ be defined as

$$V(t)=\sum_{j=1}^{J} v_j(t)=\sum_{j=1}^{J} x_j^T(t)P_j x_j(t)\ . \qquad \text{(A 1)}$$

We then evaluate the time derivative of $V(t)$ on the trajectories of Eq. (4.1) to get

$$\dot{V}=\sum_{j=1}^{J}\dot{v}_j(t)=\sum_{j=1}^{J}[\dot{x}_j^T(t)P_j x_j(t)+x_j^T(t)P_j\dot{x}_j(t)]$$

$$=\sum_{j=1}^{J}\{[\sum_{i=1}^{r_j}\sum_{l=1}^{r_j}h_{ij}(t)\,h_{lj}(t)(A_{ij}-B_{ij}K_{lj})x_j(t)+e_j(t)+\phi_j(t)+\sum_{\substack{n=1\\ n\neq j}}^{J}C_{nj}x_n(t)]^T P_j x_j(t)$$

$$+x_j^T(t)P_j[\sum_{i=1}^{r_j}\sum_{l=1}^{r_j}h_{ij}(t)\,h_{lj}(t)(A_{ij}-B_{ij}K_{lj})x_j(t)+e_j(t)+\phi_j(t)+\sum_{\substack{n=1\\ n\neq j}}^{J}C_{nj}x_n(t)\}$$

$$=\sum_{j=1}^{J}\sum_{i=1}^{r_j}\sum_{l=1}^{r_j} h_{ij}(t)\,h_{lj}(t)x_j^T(t)[(A_{ij}-B_{ij}K_{lj})^T P_j + P_j(A_{ij}-B_{ij}K_{lj})]x_j(t)$$

$$+\sum_{j=1}^{J}[e_j^T(t)P_j x_j(t)+x_j^T(t)P_j e_j(t)]+\sum_{j=1}^{J}[\phi_j(t)P_j x_j(t)+x_j^T(t)P_j\phi_j(t)]$$

$$+\sum_{j=1}^{J}\sum_{\substack{n=1\\n\neq j}}^{J}[x_n^T(t)C_{nj}^T P_j x_j(t)+x_j^T(t)P_j C_{nj}x_n(t)] \qquad \text{(A2)}$$

Based on Lemma 4.1 and Eq. (A2), we have

$$\dot{V}\le\sum_{j=1}^{J}\sum_{i=1}^{r_j}\sum_{l=1}^{r_j} h_{ij}(t)\,h_{lj}(t)x_j^T(t)[(A_{ij}-B_{ij}K_{lj})^T P_j + P_j(A_{ij}-B_{ij}K_{lj})]x_j(t)$$

$$+\sum_{j=1}^{J}\{\beta_j[e_j^T(t)e_j(t)]+\beta_j^{-1}[x_j^T(t)P_j^2 x_j(t)]\}+\sum_{j=1}^{J}\{\gamma^{-1}[x_j^T(t)P_j^2 x_j(t)]+\gamma[\phi_j{}^T(t)\phi_j(t)]\}$$

$$+\sum_{j=1}^{J}\sum_{\substack{n=1\\n\neq j}}^{J}[\lambda\; x_n^T(t)x_n(t)+\lambda^{-1}x_j^T(t)P_j C_{nj}C_{nj}^T P_j x_j(t)]$$

$$\le\sum_{j=1}^{J}\sum_{i=1}^{r_j}\sum_{l=1}^{r_j} h_{ij}(t)\,h_{lj}(t)x_j^T(t)[(A_{ij}-B_{ij}K_{lj})^T P_j + P_j(A_{ij}-B_{ij}K_{lj})]x_j(t)$$

$$+\sum_{j=1}^{J}\sum_{i=1}^{r_j}\sum_{l=1}^{r_j} h_{ij}(t)h_{lj}(t)\{\beta_j[\overline{H}_j x_j(t)]^T[\overline{H}_j x_j(t)]+\beta_j^{-1}[x_j^T(t)P_j^2 x_j(t)]\}\text{ ( from Eq. (4.5) )}$$

$$+\sum_{j=1}^{J}\sum_{i=1}^{r_j}\sum_{l=1}^{r_j} h_{ij}(t)h_{lj}(t)\,\gamma^{-1}\{[x_j^T(t)P_j^2 x_j(t)]+\gamma[\phi_j{}^T(t)\phi_j(t)]\}$$

$$+\sum_{j=1}^{J}\sum_{i=1}^{r_j}\sum_{l=1}^{r_j}\sum_{n=1}^{J} h_{ij}(t)h_{lj}(t)x_j^T(t)[\lambda\;(\frac{J-1}{J})I+\lambda^{-1}P_j C_{nj}C_{nj}^T P_j]x_j(t)$$

$$\dot{V}\le\sum_{j=1}^{J}\sum_{i=1}^{r_j}\sum_{l=1}^{r_j}\sum_{n=1}^{J} h_{ij}^2(t)x_j^T(t)\{\frac{1}{J}[(A_{ij}-B_{ij}K_{ij})^T P_j + P_j(A_{ij}-B_{ij}K_{ij})]+$$

$$\frac{1}{J}[\beta_j\overline{H}_j^T\overline{H}_j+\beta_j^{-1}P_j^2+\gamma^{-1}P_j^2]$$

$$+\lambda(\frac{J-1}{J})I+\lambda^{-1}P_j C_{nj}C_{nj}^T P_j\;\}x_j(t)+\sum_{j=1}^{J}\gamma[\phi_j{}^T(t)\phi_j(t)]$$

$$\le\sum_{j=1}^{J}\sum_{n=1}^{J}\{\sum_{i=1}^{r_j}\sum_{l=1}^{j} h_{ij}(t)h_{lj}(t)\hat{\lambda}_{inj}\}\left\|x_j(t)\right\|^2+\sum_{j=1}^{J}\gamma\left\|\phi_{upj}(t)\right\|^2\,. \qquad \text{(A3)}$$

The fuzzy interconnected system is UUB stable if the matrices $Q_{inj}$'s are negative definite. Based on Eq. (4.7) and Definition 2.1, the proof is thereby completed.

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